A study of the motion of a cavity in a rotating liquid

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Experiments are described in which a spinning tube was initially filled with water and closed at both ends; when the water had acquired uniform angular velocity the tube was suddenly opened at one end and hence emptied by centrifugal action, so that a cavity progressed along it towards the far end. The velocity of the cavity was found to be steady and proportional to the speed of rotation over the range tested, which confirmed the supposition that gravity and viscosity had insignificant effects on the cavity motion. Contrary to expectation, since the cavity velocity seemed to be too large for it to occur, the 'Taylor phenomenon' was observed in the liquid ahead of the cavity; that is, the motion generated by the invasion of the cavity extended over a continually lengthening region beyond it.

The theoretical discussion in § 4 explains several features of the experiments satisfactorily, although the complete analytical problem has so far proved insoluble.

1. Introduction

The experiments to be described were conceived originally as a possible demonstration of principles that had come to light in a theoretical study of the vortex breakdown phenomenon (Brooke Benjamin 1962). The experimental observations in fact failed to reveal the simple behaviour anticipated, and they posed a theoretical problem far more difficult than the one presupposed. Nevertheless, even though we have been unable to carry out a satisfactory analysis, our experimental results may be worth presenting for their own interest. Some relevant theoretical points will be summarized in the penultimate section of the paper, and it is hoped that these may at least clear the ground for some subsequent attempt to provide a complete theory.

In the experiments a long tube was filled with water and spun steadily about its axis until the water acquired 'solid-body rotation'. The tube was then opened at one end, and since the centripetal acceleration of the water was much larger than g the tube emptied axisymmetrically. In the first stage of the emptying process a cavity progressed rapidly along the tube, taking up the volume left by water discharged from the open end. Dimensional reasoning suggested that, if the tube were long enough and the rotation were rapid enough for gravity and

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viscosity to be unimportant, the velocity U of the cavity would become steady at a value proportional to ΩR, where Ω is the angular velocity of the tube and R its internal radius; and this expectation was indeed borne out closely by the velocity measurements. But since the ratio $U/\Omega R$ (the 'inverse Rossby number') was unknown initially, there was no way of foretelling the character of the flow as regards the well-known general distinctions depending on Rossby number (Squire 1956, § 1.3).

To appreciate this aspect of the experiments, it is helpful to review the possibilities presented *a priori* by an ideal-fluid model. Let 'type A' designate a flow that is steady in a reference frame moving with the cavity and is undisturbed far ahead of it, and 'type B' a flow in which a continually lengthening column of fluid is pushed ahead of the cavity (i.e. in the manner of the phenomenon observed by Taylor (1922) and Long (1953) in their experiments where a solid body was moved slowly along the axis of a uniformly rotating liquid). Also let $c_0 = 0{\cdot}522\Omega R$ denote the maximum group velocity of infinitesimal waves (see § 4). Then the possibilities can be listed as follows:

(1) $U > c_0$: flow of type A;

(2) $U < c_0$, but $U/\Omega R$ not distinctly small: either (a) flow of type A, or (b) flow of type B;

(3) $U \ll \Omega R$: flow of type B.

Following the history of our investigation, we at first expected a flow of type A, which would have given us a simple analogy with the vortex breakdown phenomenon, and this possibility remained in view when the cavity velocity had been measured and found to be in the range (2). We then discovered a theoretical argument, which is explained in § 4, showing that a flow of type A cannot occur past a cavity at constant pressure. (Some further analysis bearing on this point is contributed by Mr L. E. Fraenkel in an appendix to this paper.) Finally, after an adequate flow-visualization technique had been developed, it was established that in fact the possibility (2b) is realized.

As the rotating tube was horizontal in the experiments, the obvious criterion for assuming gravity to have negligible effect on the cavity was that $\Omega^2 R/g \gg 1$. For the observations recorded below, this ratio ranged from 41 to 138, these values presumably being high enough to justify the assumption.

2. Experimental apparatus and procedure

A Perspex tube of length 65 in. and bore 2 in. was mounted in five ball-bearings which were fixed to a steel channel clamped to a massive base. The tube was driven by an electric motor through a variable-ratio gearbox and a belt and pulley system. The tube was sealed at one end, and at the other a stopper was fitted which could be removed while the tube was rotating. The stopper was designed for quick removal when the tube was completely filled with water, and because of the effective incompressibility of water it was necessary to admit a small volume of air before it could be withdrawn. In order to minimize disturbances to the flow the stopper, shown in figure 1, was designed to admit the air near the axis of the tube.

The body of the stopper carried an O ring at its circumference which formed a sliding seal at the mouth of the tube. Six holes drilled through the stopper near the axis were normally covered by a disk attached to a rod passing through the centre of the stopper. The disk was held against an O ring face seal by a spring. By pulling the rod to the right the holes were uncovered and air was admitted as the stopper was forced out of the tube by the spring. Tension was applied to the rotating rod through the loose wheel at its right-hand end.

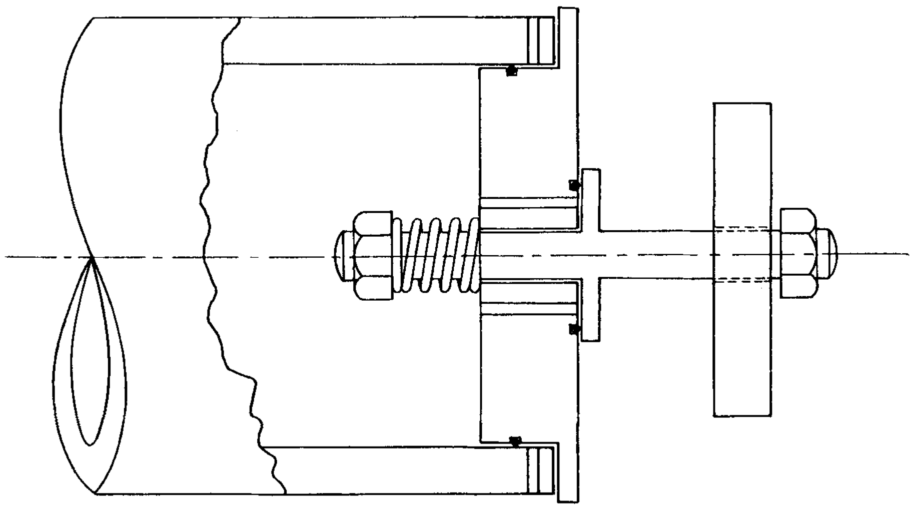

FIGURE 1. Detail of stopper.

It was necessary to incline the steel channel carrying the tube to fill it with water. The final addition of water was made with the stopper in place but withdrawn sufficiently to uncover the pair of holes drilled through the tube wall. Water was forced in through the lower hole, and when all the air had escaped by the upper hole the stopper was pushed home. The excess water escaped through the six holes after its pressure had lifted the disk from its seat. When stoppered and secured to the base with its axis horizontal, the tube was driven at a constant speed Ω (rad/sec). After a pause of a few minutes to allow the water to acquire solid-body rotation through the action of viscosity, the stopper was removed. The motion then observed can be described as follows.

Water escaped through the open end of the tube and was replaced by atmospheric air, which filled an axisymmetric cavity or core progressing steadily along the tube. In the annulus of water surrounding the cavity the net flow towards the open end accommodated the rate of volume displacement by the cavity; but a layer of water next to the cavity, being bounded by a surface at constant pressure, had to be carried along with it into the tube. The speed of the cavity along the tube and the velocity field in the water ahead of the cavity were observed for various speeds of rotation.

Ciné-photographs of the motion were taken by transmitted light from a diffuse source. The general arrangement is shown in figure 2, where the position of the ciné camera at about 6 ft. from the tube is indicated by the arrow. A length scale and a disk rotating at a known speed to indicate time were photographed with the tube. Parallax was reduced by the plane mirror system shown in the figure; the tube was photographed in three sections, one directly and two by reflexion, which were made to appear one above the other by tilting the mirrors slightly from the vertical. Photographs were taken at a rate of approximately 80 frames/sec. With the position of the cavity recorded at a succession of known time intervals, its velocity was directly deducible.

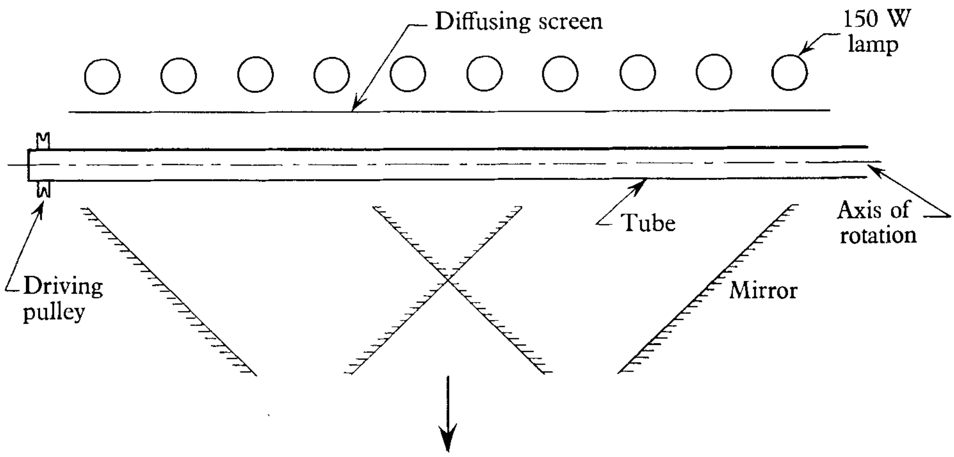

FIGURE 2. General arrangement for ciné-photography.

In order to observe the flow ahead of the cavity, a visualization technique was necessary. Three fine platinum wires with an electroplated skin of tellurium were positioned along diameters of the tube at 18, 32 and 47 in. from the open end. Tellurium evolves a black dye composed of a suspension of fine particles of the metal when it is the cathode of an electrolytic cell. The electrochemical reaction which takes place at the cathode to produce the dye has been described by Wortmann (1953). The tellurium-plated wires were connected to three slip-rings and an additional electrode (anode) was connected to earth through one of the bearings. Just before the tube was opened dye was released for about 1 sec by applying a potential difference of 200 V between the slip-rings and earth. The fluid particles originally situated along the selected diameters of the tube were thus marked and their subsequent motion was observed from the ciné-film record.

[An alternative method of measuring the cavity velocity was tried which may be worth mentioning. The time of passage of the cavity between two stations along the tube was measured electronically. At each station a pair of slender insulated probes with bare tips were aligned along a diameter, leaving a small separation between the tips at the axis of the tube. Slip-rings and the water between the tips completed electric circuits, and, when these were broken by the cavity, timing signals were obtained. High-frequency alternating currents were used to minimize effects of polarization of the probes. The method proved unreliable because of breakdown in the thin layer of insulating varnish around the probes, and was abandoned in favour of the ciné-photographic method.]

Figure 3 (plate 1) shows typical photographs of (*a*) the cavity and (*b*) the tellurium dye trace. Certain features of these photographs will be commented upon in § 4.

3. Experimental results

The ciné-films were examined frame by frame and the positions of the cavity nose and the marked water situated on the tube axis were noted. This information was used to prepare curves of displacement versus time, a typical set of which is presented in figure 4. In the figure the slope of the curve joining the

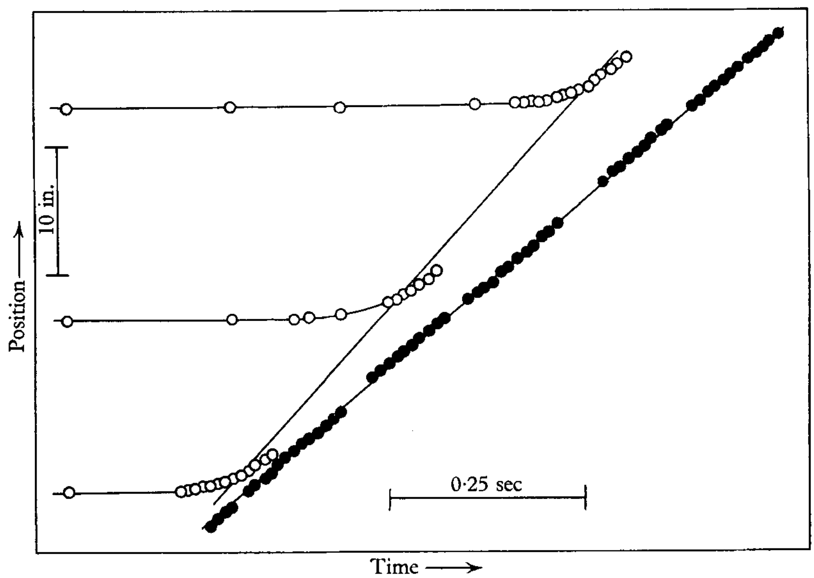

FIGURE 4. Typical results obtained from ciné-photographs, showing displacement *vs* time: ●, position of nose of cavity; ○, axial position of tellurium dye trace.

solid circles gives the cavity speed U, which is seen to be substantially constant. A straight line has been drawn through the points to emphasize their deviation; the systematic nature of the deviation suggests that it may have been due to parallax, which changed in sign abruptly when the cavity passed from one to the next of the three phases in which it was photographed (see figure 2). The gaps in the series of points are due to the passage of the cavity behind the bearings and slip-rings.

The sets of points indicated by open circles in figure 4 show the progress of the marked water in three places along the axis. The intensity of the dye traces decreased rather quickly with time, owing to diffusion of the dye, and consequently the points at the right-hand ends of the curves are less reliable than the earlier ones. It appears from the figure that the three curves are similar in form, but their scales show a slight though steady increase from the first to the third. A line has been drawn joining the three points at which the instantaneous velocity of the marked fluid elements was $0{\cdot}5U$. The slope of this line gives the speed C at which an axial water velocity of $0{\cdot}5U$ propagated along the tube.

Estimates of U and C taken from all the diagrams like figure 4 are presented in figure 5. It appears that the dimensionless parameters $U/\Omega R$ and $C/\Omega R$ were essentially constant in the range of the experiments, their values being about 0·38 and 0·48, respectively. The radius of the cavity was estimated to be $0{\cdot}5R$.

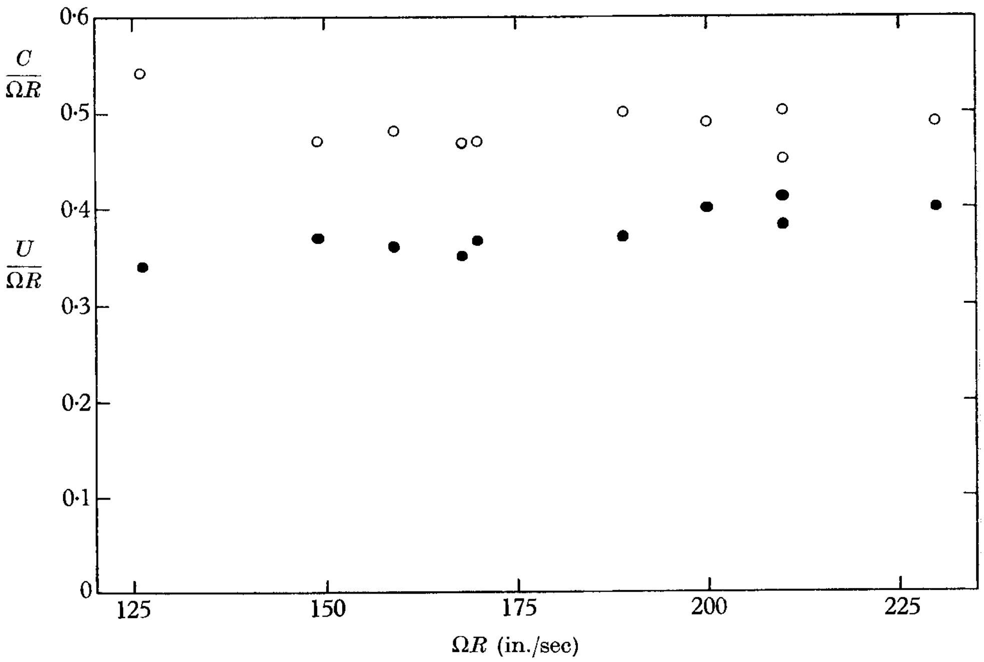

FIGURE 5. Dependence of U and C on speed of rotation: ●, $U/\Omega R$; ○, $C/\Omega R$.

4. Theoretical discussion

When the experiments finally established the overall character of the flow, the theoretical problem was seen to be much more difficult than had been expected. Although little has been achieved so far in the analysis of the exact problem, it nevertheless seems useful to review some points of theory that have been considered in relation to the experiments.

A related problem of steady motion

We first refer to the problem indicated by figure 6. A semi-infinite body of revolution is fixed symmetrically in a tube of circular cross-section, the radii of the tube and the cylindrical part of the body behind the nose being R and λ, respectively. An inviscid fluid with constant density ρ flows steadily along the tube, and far ahead of the body it has uniform axial velocity U and angular velocity Ω. In the annular space far downstream the stream-surfaces become cylindrical asymptotically (i.e. it is assumed standing waves do not arise), and the aim in view is to find the hydrodynamic drag on the body by consideration of a momentum balance between this and the original cylindrical flow. Then if an example were forthcoming in which the pressure on the body surface far downstream were equal to stagnation pressure on the axis upstream and the drag calculated this particular way were zero (or negative, see below), one might suppose the solution to be applicable to the case of a cavity advancing at velocity U into a rotating fluid as in our experiments.

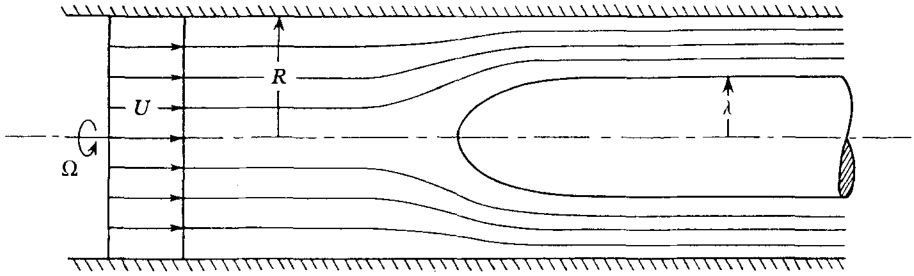

FIGURE 6. Steady swirling flow past axisymmetric body, the fluid upstream having uniform axial velocity U and angular velocity Ω.

When the shape of the body is specified, the flow will be determined completely by $U/\Omega R$ and λ/R, which are the only independent dimensionless parameters of the physical situation; but it appears that steady flows of the type depicted in figure 6 are realizable only over a certain range of these parameters. The value of $U/\Omega\lambda$ is probably the main criterion, and certainly if it is very small the 'Taylor phenomenon' will occur, producing a continually lengthening column of stagnant fluid ahead of the body (cf. Squire 1956, § 1.3). When the column has extended to distances much greater than R the flow about the body will tend to become steady; but then a new, and far more difficult, problem of steady motion is presented in that the axial and angular velocities ahead of the body are no longer uniform.

That this phenomenon is encountered in the cavity problem appears surprising in view of the experiments by Taylor (1922) and Long (1953), the results of which rather suggest that the value $U/\Omega\lambda \simeq 0{\cdot}8$ observed here would be too large for it to occur. For instance, Long made his experiments with a body having a hemispherical nose and conical tail, its radius a being about one-third of the radius of the tube along which it was moved at velocity U; he found that fluid near the axis was pushed ahead of the body only when $U/\Omega a$ was less than about $0{\cdot}2$. Taylor gave $1/\pi$ as a rough estimate of the corresponding value for a sphere of radius a.† In the first stages of the present experimental work these antecedents in fact led us to disregard the possibility of the Taylor phenomenon occurring, and a search for it was made only after the discovery of a theoretical argument, which will now be outlined, indicating that steady flow of the kind in question is impossible past a cavity.

† For a sphere in an unbounded rotating fluid, however, Stewartson (1958) has suggested that the Taylor phenomenon may arise at values of $U/\Omega a$ as high as about $\frac{2}{3}$. He showed that in its absence the drag on the sphere becomes unreasonably large when $U/\Omega a$ falls below this limit.

For a steady axisymmetric flow originating as described above, the stream function ψ satisfies the linear equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{4\Omega^2}{U^2}\psi=\frac{2\Omega^2r^2}{U^2}, \tag{1}$$

in which x and r are axial and radial co-ordinates (cf. Squire 1956, equation (32)). The solution $\psi = \frac{1}{2}Ur^2$ of (1) represents, of course, the original cylindrical flow. For the flow in the annular space far downstream, another solution independent of x is needed and this can be expressed in the form

$$\psi = \tfrac{1}{2}U\left\{r^2+\frac{r}{\kappa}Z_1(2\kappa r)\right\}, \tag{2}$$

where $\kappa = \Omega/U$ and where Z_1 denotes a linear combination of the first-order Bessel functions of the first and second kinds, thus implying two disposable numerical constants. Corresponding to (2) the axial velocity is

$$u=\frac{1}{r}\frac{d\psi}{dr}=U\{1+Z_0(2\kappa r)\}. \tag{3}$$

The kinematical boundary conditions on this flow are that $\psi(R) = \frac{1}{2}UR^2$ and $\psi(\lambda) = 0$, which require that

$$Z_1(2\kappa R)=0 \quad\text{and}\quad Z_1(2\kappa\lambda)=-\kappa\lambda. \tag{4}$$

The two numerical constants in (2) are fixed by (4), so that with U, Ω, R and λ specified the solution is determined completely. [With regard to the cavity problem, the idea in view is that amongst the class of such solutions given by varying κ and λ parametrically there might be one satisfying the appropriate pressure condition at $r = \lambda$ and overall momentum condition; as will be explained later, the additional boundary condition is that $u(\lambda) = 0$ and so $J_0(2\kappa\lambda) = -1$, which combined with (4) implies that $J_2(2\kappa\lambda) = 0$.]

The drag D on the 'half body' may be defined as follows (cf. Prandtl & Tietjens 1957, §78). Let p denote the pressure in the fluid and p_λ its (constant) value at the body surface $r = \lambda$ far downstream. Then D is given by the integral of $p-p_\lambda$ over the projected area of the nose. But, by a momentum balance for the steady flow, the integral of pressure over the nose must equal the difference in 'flow-force' (i.e. pressure force plus momentum flux) between two sections through the flow, the first far upstream and the second far downstream from the nose. Therefore the quantity

$$D' = 2\pi\left\{\int_0^R (p+\rho U^2)r\,dr-\int_\lambda^R (p+\rho u^2)r\,dr\right\}-\pi\lambda^2p_\lambda, \tag{5}$$

where the integrals are evaluated upstream and downstream respectively, must be equal to D for the momentum balance to be satisfied. Since dissipation is assumed absent the stagnation pressure at $r = \lambda$ far downstream is the same as the value on the axis upstream, say P_0, and so we have that $p_\lambda = P_0-\frac{1}{2}\rho u_\lambda^2$. In the first of the two integrals on the right-hand side of (5) the pressure is expressible as $p = P_0-\frac{1}{2}\rho U^2+\frac{1}{2}\rho(\Omega r)^2$, which follows from the facts that in any cylindrical flow $dp/dr = \rho w^2/r$, where w is the velocity of swirl, and that $w = \Omega r$ upstream; hence the integration can be performed immediately. Furthermore, since the circulation $2\pi rw$ remains constant along each stream surface, we have $rw = 2\kappa\psi$ and hence, for substitution in the second integral,

$$p = p_\lambda + 4\rho\kappa^2 \int_\lambda^r \frac{\psi^2}{r^3} dr.$$

It remains only to substitute (2) and (3) in order to find D' explicitly in terms of the given physical parameters, and clearly P_0 will cancel from the result.

Now, if a cavity at constant pressure were to replace the solid body of figure 6, D would obviously be zero according to our basic definition. For the present flow model to be precisely applicable, the appropriate solution (2) should therefore make the right-hand side of (5) vanish. If (5) happened to give a negative value of D', however, the possibility of steady flow past a cavity could not be ruled out immediately. Indeed, precisely this outcome was expected by the authors at first, while the problem was still conjectured to depend on the same principle as vortex breakdown. A negative value of D' according to (5) would imply that a *cylindrical* flow downstream had an excess of flow-force above the level essential to a steady-state momentum balance with $D = 0$; but it would be plausible that a balance might still be brought about by wave formation, the required reduction in flow-force then corresponding to the 'wave resistance' of a stationary wave-train developed along the cavity (cf. Brooke Benjamin 1962, §§ 1, 4.6).†

The impossibility of either case conjectured above is demonstrable in the following way. Treating the right-hand side of (5) as outlined below the equation, performing the various integrations and using the boundary conditions (4) to reduce the integrated terms, one obtains the result

$$D' = \tfrac{1}{2}\pi\rho\lambda^2(u_\lambda^2 - Uu_\lambda + \tfrac{1}{2}\Omega^2\lambda^2). \tag{6}$$

This holds for all steady flows of the kind illustrated in figure 6. In the case of a cavity at constant pressure, however, the existence of the stagnation point at its forward end implies that the fluid must be at rest, relative to the stagnation point, everywhere on its surface (i.e. $p_\lambda = P_0$ and so $u_\lambda = 0$).‡ With $u_\lambda = 0$

† The general principle in view here may be stated as follows: Whenever a hypothetical case of *streamwise uniform* flow, determined by mass and energy conservation, gives a flow-force value too large to be physically admissible in a steady state, there may be a realizable steady flow which is formed by the superposition of periodic waves on the hypothetical one, with perhaps some slight dissipation of energy as an additional requirement. The categorical fact underlying any such possibility is that the property of flow-force reduction (i.e. positive wave resistance) is inherent in any natural periodic-wave system which becomes steady after developing *downstream* from its originating agency; thus, for infinitesimal waves in particular, this property is concomitant with their group velocity being directed downstream. This principle serves to explain several practically interesting examples of wavy flow, notably undular hydraulic jumps and mild vortex breakdowns; it also applies to the filamentary vapour cavities that are often formed in the tip vortices shed from ships' propellers (Brooke Benjamin, 1962, §4.7).

‡ In figure 3(*a*), plate 1, the 'ventilated' cavity seen attached to part of the wire spanning the tube shows that the fluid there was moving forward with the main cavity.

equation (6) gives $D' > 0$ necessarily, thus contradicting the condition $D' \leqslant D = 0$ demanded by the steady-state momentum balance.

It must be asked whether the foregoing argument establishes conclusively that steady flow of the type A defined in § 1 is impossible. The authors believe that it does indeed stand as a proof, inasmuch as they presume the truth of the general principle explained in the long footnote above—or rather the truth of its converse that a corresponding parallel flow is always essential to the existence of a physically realizable wavy flow. In certain ways, however, this somewhat oblique line of argument is less immediately convincing than an alternative to which reference will be made below. To compare the respective advantages of the two, and justify our present option, the following questions need to be distinguished: (i) For the idealized model of the fluid and cavity system, does a mathematical solution of type A exist subject to velocities being finite everywhere? (ii) Even if the answer to (i) is negative, is it still not possible that a flow of type A might be realized in practice when the system is slightly dissipative?

Although the present argument gives, we believe, a decisive answer to (i), it is clearly less satisfactory than one dealing directly with the possibility of a wave-shaped cavity and so avoiding dependence on the 'wave resistance principle'—of which no formal proof is yet available for reference. While discussing this theoretical problem with us, Mr L. E. Fraenkel pointed out such an alternative form of argument, and he has kindly supplied a presentation of it which is given in the Appendix to this paper. His analysis confirms rigorously that the answer to (i) is negative.

On the other hand, the special advantage of the present argument is that it also answers question (ii). The essential principle applies equally well to slightly dissipative systems as to idealized ones, and there are several precedents for supposing that it generally provides a more powerful means of determining the possible behaviour of a real flow than does an *a priori* deduction of properties according to a frictionless theoretical model.†

While the analysis so far considered has regrettably little direct bearing on the flow observed experimentally, at least one of the ideas presented seems vital to the prospects for a satisfactory theory of the cavity motion. Namely, the principle just discussed seems very likely to apply to the quasi-steady flow which actually develops in the vicinity of the cavity. Behind the front of the disturbance propagating ahead of the cavity, the flow appeared in the experiments to become steady and cylindrical; moreover, as will be noted later, its probable analytical representation is fairly simple. A problem of steady relative motion can therefore be posed easily enough, but the equation corresponding to (1) for the stream function is awkwardly non-linear in this instance and its solution

† Consider, for example, the case of slightly supercritical flow in an open channel. Assuming conditions of *exact* energy and flow-force conservation, one may prove that no steady disturbance can exist extending indefinitely far downstream; in fact the only possible steady disturbance under these conditions is the solitary wave (see Brooke Benjamin & Lighthill 1954). But the principle in question indicates, correctly, that disturbances in the form of wave-trains are realizable when a small amount of energy is dissipated, thus accounting for the 'undular bores' which readily arise in practice on slightly supercritical streams.

presents very formidable difficulties. It would probably be hopeless to attempt a solution allowing directly for the possibility of a wave-shaped cavity, and the simpler approach based on the aforementioned principle appears much more promising. That is to say, one might best proceed on the same lines as the preceding discussion, deriving the solution for a hypothetical cylindrical flow far downstream and then examining the momentum balance. If, as presumably would happen, this flow turned out to have an excess of flow force, the true flow could be represented by superposing waves on the cavity. Our photographs of the cavity in fact revealed the presence of waves (see figure 3 (*a*), plate 1).

The motion ahead of the cavity

We next examine the properties of the column of fluid observed in the experiments to be pushed ahead of the cavity. The development of the Taylor phenomenon with time has been analysed by Stewartson (1952) in the particular case of a sphere moved *slowly* along the axis of rotation; but his theory depends on linearization of the equations of motion and so is inadequate for the present problem, in which the velocity perturbations in the fluid are not everywhere small in comparison with the initial velocity of rotation. At the outskirts of the column, however, there must be a region where the perturbations are still small enough for a linearized approximation to apply, and on the basis of this fact certain properties can be deduced fairly easily.

For small axisymmetric disturbances from the original state of uniform angular velocity Ω, the axial velocity $u(x, r, t)$ satisfies

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+4\Omega^2\frac{\partial^2 u}{\partial x^2}=0 \tag{7}$$

(cf. Squire 1956, equation (20) for the stream function), and the boundary conditions on u are

$$\partial u/\partial r = 0 \quad \text{at} \quad r=0 \quad \text{and} \quad r=R. \tag{8}$$

(The reason for (8) is that no circumferential component of vorticity can arise on the axis or the wall, where obviously the radial velocity component must also be zero.) We shall take the direction of x to be towards the undisturbed fluid ahead of the advancing column.

Solutions of (7) and (8) representing travelling waves exist in the form

$$u = AJ_0(kr)\,e^{i(\omega t-\alpha x)}, \tag{9}$$

provided ω, α and k are related by

$$\omega^2(\alpha^2+k^2) = 4\Omega^2\alpha^2 \tag{10}$$

and kR is a zero of J_1. To find the behaviour of the head of the column at large times, only the solutions corresponding to the first (positive) zero $kR = 3{\cdot}832$ need be considered since these solutions have the largest phase velocity $c = \omega/\alpha$. Accordingly (10) gives

$$\omega = \frac{c_0\alpha}{(1+0{\cdot}0681\alpha^2R^2)^{\frac{1}{2}}}, \tag{11}$$

where $c_0 = 2\Omega R/3{\cdot}832 = 0{\cdot}522\Omega R$. Note that c_0 is the maximum phase velocity and maximum group velocity $d\omega/d\alpha$ for this class of travelling waves (cf. Fraenkel 1956, §4.3).

It must be recognized that the foregoing results also account for a class of solutions with real exponential dependence on x and t. Putting $\alpha = -im$ and $\omega = -in$ and taking m and n to be real and positive, we obtain

$$u = AJ_0(3{\cdot}83r/R)\exp\{-m(x-ct)\} \tag{12}$$

with

$$c = \frac{n}{m} = \frac{c_0}{(1-0{\cdot}0681m^2R^2)^{\frac{1}{2}}}. \tag{13}$$

This represents the outskirts of a disturbance that maintains its form while advancing at a velocity greater than c_0, and the question arises whether our column might have such a character. The possibility of a steady wave form can be ruled out, however, after consideration of the equivalent problem of steady motion on the basis of equation (1). There readily appears to be no solution which, in the absence of any obstacle displacing fluid inside the boundary $r = R$, has the required properties of tending to the undisturbed flow for $x \to \infty$ and tending to another flow with finite velocities for $x \to -\infty$.

On the other hand, the solution (12) will apply ahead of an axisymmetric obstacle moving faster than c_0 or, with obvious modifications, it will apply upstream in a corresponding steady flow such as in figure 6, where now $U > c_0$. And since real exponentials are the only small-amplitude solutions possible under this condition, it follows that a situation such as depicted in figure 6 must eventually become steady if $U/\Omega R > 0{\cdot}522$; that is to say, an incessant dispersive process is impossible in front of the obstacle since no periodic component like (9) has a large enough group velocity to remain there. Thus we have a sufficient condition for non-occurrence of the Taylor phenomenon. [It does not, of course, follow that the Taylor phenomenon will necessarily occur if $U/\Omega R < 0{\cdot}522$; for in this case solutions of the real exponential type are still obtainable by assigning kR to the higher zeros of J_1.]

Having established that the motion at the outskirts of the column is unsteady, we may conclude that the appropriate solution of (7) at large t is expressible in the form†

$$u = AJ_0(3{\cdot}83r/R)f(x,t), \tag{14}$$

where $f(x,t)$ is a Fourier integral over a spectrum of wave components as in (9), with ω related to α through (11). There is no way of determining the composition of the spectrum on the basis of the present linearized theory, but the following hypothetical case serves to illustrate the general character of the motion.

We consider the form of $f(x,t)$ corresponding to the initial values

$$f(x,0)\begin{cases} = 1, & x < 0, \\ = 0, & x > 0. \end{cases}$$

† As it obviously must, this axial-velocity distribution gives a zero net flow through any cross-section, the forward flow near the axis being balanced by a return flow through the outer part of the cross-section (in fact through the annulus $0{\cdot}628R < r \leqslant R$).

This is
$$f(x,t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{i(\omega t-\alpha x)}\frac{d\alpha}{\alpha}, \tag{15}$$

with the path of integration indented under the origin. There is some support for this choice of Fourier integral in the fact that, for all $t \geqslant 0$, it gives $f \to 1$ behind the wave front, whereas with $f = 1$ the expression (14) is an asymptotic solution of the full non-linear equations of motion under conditions where the x- and t-derivatives tend to zero. Thus the solution may accurately describe the motion in the cylindrical part of the column observed behind the dispersing front. In this part, moreover, an indication of the experiments was that fluid on the axis became stationary relative to the cavity, implying that the constant A may equal the cavity velocity $0{\cdot}38\Omega R$. In other words, our solution of the linearized equations of motion has the incidental property of matching precisely to the flow well inside the column, even though in the leading part it is reliable only at the extremities where $f \ll 1$.

According to (11) the front of the disturbance will eventually be composed from waves with group velocities near c_0, i.e. with α small; and so to find an asymptotic approximation to (15) for large t we may take the second approximation to (11) for α small, thus

$$\omega = \alpha c_0(1-0{\cdot}034\alpha^2R^2). \tag{16}$$

[It is noteworthy that a similar situation is presented by the linearized theory of long waves in shallow water (Jeffreys & Jeffreys 1946, § 17.09).] Introducing

$$\zeta = (0{\cdot}102R^2c_0t)^{\frac{1}{3}}\alpha, \quad z = (x-c_0t)/(0{\cdot}102R^2c_0t)^{\frac{1}{3}}, \tag{17}$$

we then obtain
$$f \sim F(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\exp\{-i(z\zeta+\tfrac{1}{3}\zeta^3)\}\frac{d\zeta}{\zeta}$$
$$= \int_z^{\infty}\mathrm{Ai}\,(z)\,dz, \tag{18}$$

where
$$\mathrm{Ai}\,(z) = \frac{1}{\pi}\int_0^{\infty}\cos(z\zeta+\tfrac{1}{3}\zeta^3)\,d\zeta$$

is the Airy integral.

This result represents an advancing wave-form which keeps a constant value at $x = c_0t$, but whose length scale measured about this point increases with time like $t^{\frac{1}{3}}$. Towards negative z the function $F(z)$ defined by (18) oscillates with steadily decreasing amplitude about its asymptotic value of 1; but this feature is probably spurious, owing to the failure of the linearized approximation in the region where f is not small. Oscillations were not observed at the head of the column in the experiments, although we are unable to say with certainty that they were absent, and so it seems possible that the actual wave-form makes a smooth transition between the outskirts, where its initial rise may be described accurately by (18), and the cylindrical flow inside the column. At the outskirts an approximation to (18) by steepest descents gives

$$f \sim \frac{1}{2\pi^{\frac{1}{2}}}\frac{1}{z^{\frac{3}{4}}}\exp(-\tfrac{2}{3}z^{\frac{3}{2}}), \tag{19}$$

which is fairly accurate for z greater than about 1 and which shows how the disturbance diminishes smoothly to zero towards positive z.

While the tentative nature of these results needs to be emphasized, they at least suffice to illustrate the main theoretical conclusion that the front of the column advances with the characteristic speed c_0 while gradually dispersing. This was borne out reasonably well by the experimental observations. Tracing the progress of the point at which fluid on the axis acquired a velocity half that of the cavity, we estimated its speed of propagation as $C = 0{\cdot}48\Omega R$. The speeds of points further ahead associated with smaller values of u appeared to be somewhat greater than this, but they were difficult to estimate and the present C is sufficiently close to c_0 for general confidence in the theoretical interpretation. The difference of 8 % might anyway be accountable to an effect of the wires which spanned the flow. Some slight indication of the predicted dispersion of the front of the column was forthcoming from the diagrams exemplified by figure 4, but the data were inadequate for a definite appraisement in this regard. The axial-velocity distribution in the column could be estimated from the displacement of the dye from its initial position along a diameter, and the observations confirmed the theoretical distribution $J_0(3{\cdot}83r/R)$. It should be noted, however, that figure 3(*b*) (plate 1) gives a distorted view of this distribution, owing to the curved walls of the tube.

5. Conclusion

It may be useful now to summarize the overall picture established by the experiment and rationalized at least in part by the preceding theoretical discussion. When a tube initially filled with liquid having uniform angular velocity is opened wide at one end, the liquid flows out by centrifugal action and, if the rotation is rapid enough for gravity to be unimportant, the flow is approximately axisymmetric. Provided the tube is long enough the bubble progressing along it may eventually evolve a fluid motion that does not depend on conditions at the open end: that is, the situation at the forward end of the bubble determines its speed and cross-sectional area, and hence the rate of flow of liquid out of the tube, so that the conditions at outlet must adjust to this independently determined flow rate. The speed of the bubble was observed to be about $0{\cdot}38\Omega R$, and its cross-sectional radius about $0{\cdot}5R$.

When the bubble acquires its ultimate steady speed the fluid motion ahead does not become steady everywhere, there being a manifestation of the 'Taylor phenomenon'. The region in which fluid is set in axial motion ahead of the bubble continually lengthens, although behind the slowly dispersing front of this region the motion does become steady relative to the bubble. From the measurements by the dye technique the velocity of the axial point where the flow velocity becomes half the bubble velocity was estimated as $0{\cdot}48\Omega R$, which compares well with the theoretical propagation velocity $c_0 = 0{\cdot}522\Omega R$ characteristic of the front.

No evidence of waves in the region ahead of the bubble was noticed, but more or less regular undulations on the bubble surface were generally observed in photographs such as figure 3 (*a*), plate 1. It seems quite likely that these waves were an essential feature of the flow past the bubble, and not merely an effect of vibrations in the apparatus.

The conclusion that the motion of the bubble becomes independent of the precise conditions near the open end of the tube seemed fairly well supported by the observations, and it appears quite reasonable on intuitive physical grounds; but obviously certain reservations must be made in this regard. When the flow out of the tube is obstructed in some way, say by an annular weir extending around the circumference of the tube, the present contention is that the disturbance created will not, if it is reasonably small, affect the rate of discharge which is determined by events far down the tube. But clearly the flow can be throttled by any sufficiently large obstruction at the end. The influence of end conditions on flows of the present type might be worth further investigation.

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APPENDIX

The non-existence of a suitable solution for inviscid flow of type A

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A theoretical argument in § 4 strongly suggests, but does not quite prove, that the flow investigated could not be of 'type *A*' (i.e. steady in a reference frame moving with the cavity and undisturbed far ahead of it). In the course of a recent correspondence with Dr Brooke Benjamin I noticed that the non-existence of such a solution can be proved without any detailed assumption about the nature of the boundary and the flow far downstream (which might be wave-like). Accordingly the authors have invited me to present this proof here. Its relation to their argument is discussed in § 4.

Let the boundary of the cavity (figure 6) be denoted by $r = \lambda(x)$, $x \geqslant 0$. We do not assume that its slope, $d\lambda/dx$, and the radial velocity vanish far downstream. We seek $U/\Omega R$, λ and a stream function $\psi = \frac{1}{2}Ur^2 + \chi(x, r)$ satisfying the differential equation (1) and the following conditions:

(i) There is no disturbance far upstream; thus

$$\chi, \chi_x, \chi_r \to 0 \quad \text{as} \quad x \to -\infty. \tag{A 1}$$

(ii) By conservation of mass

$$\chi(x, R) = 0, \quad \chi(x, \lambda) = -\tfrac{1}{2}U\lambda^2. \tag{A 2}$$

(iii) The velocity is bounded everywhere in the fluid domain. (There is no possibility of isolated singularities for a cavity flow.)

(iv) There is a stagnation point at the nose of the cavity. Since on $r = \lambda$ the pressure is constant and the circulation $2\pi wr$ zero, and since the stagnation pressure $P(\psi)$ is constant on any stream surface, we have

$$\text{on} \quad r = \lambda(x), \quad p = P(0) \equiv P_0, \quad u \equiv U + \frac{1}{r}\chi_r = 0, \quad v \equiv -\frac{1}{r}\chi_x = 0. \tag{A 3}$$

(v) The radius of the cavity is non-vanishing behind the nose; say

$$\lambda(x) \geqslant \lambda_0 > 0 \quad \text{for} \quad x \geqslant x_0 > 0. \tag{A 4}$$

(This could be replaced by a weaker, but more complicated, condition.)

We proceed to show that no solution with these properties exists. The differential equation for χ is

$$\mathrm{L}\chi \equiv \left(\frac{1}{r}\chi_r\right)_r + \frac{1}{r}\chi_{xx} + \frac{4\Omega^2}{U^2 r}\chi = 0. \tag{A 5}$$

Let C be a closed curve in the (x, r)-plane $(r \geqslant 0)$, bounding a region S of the fluid domain. Then by Stokes's theorem

$$\int_C \left\{-2\chi_x\chi_r\frac{dx}{r} + \left[\chi_x^2 - \chi_r^2 + \frac{4\Omega^2}{U^2}\chi^2\right]\frac{dr}{r}\right\} = 2\iint_S \chi_x \mathrm{L}\chi\, dx\, dr = 0, \tag{A 6}$$

which is a mathematical statement of the momentum principle considered in § 4 Also, integration by parts and (A 5) yield for a path x = const.,

$$\int \chi_r^2 \frac{dr}{r} = \frac{1}{r}\chi_r\chi + \int\left(\chi_{xx} + \frac{4\Omega^2}{U^2}\chi\right)\chi\frac{dr}{r}. \tag{A 7}$$

Assume that the solution for the cavity exists. Choose C to consist of arcs along the cavity boundary $(0 \leqslant x \leqslant x_2,\ r = \lambda)$, along a radial line downstream $(x = x_2 > 0,\ \lambda \leqslant r \leqslant R)$, along the pipe wall $(x_1 \leqslant x \leqslant x_2,\ r = R)$, along a radial line upstream $(x = x_1 < 0,\ 0 \leqslant r \leqslant R)$, and along the axis $(x_1 \leqslant x \leqslant 0,\ r = 0)$. Let $x_1 \to -\infty$. Only the first two arcs contribute to the line integral in (A 6); hence inserting the boundary conditions for the cavity, and using (A 7) for the contribution from $x = x_2$, we obtain

$$\tfrac{1}{4}\Omega^2\lambda^4(x_2) + \int_{\lambda(x_2)}^{R}(\chi_x^2 - \chi\chi_{xx})\frac{dr}{r} = 0. \tag{A 8}$$

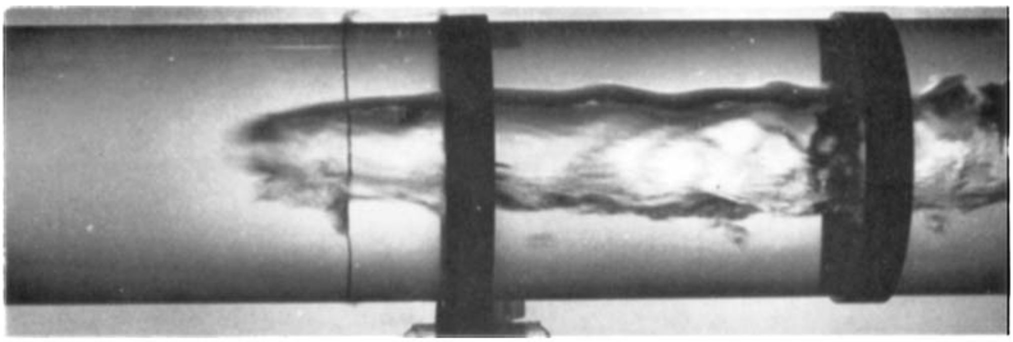
If the flow downstream is assumed to be parallel ($\chi_x \to 0$ for $x \to \infty$), the last integral vanishes, leaving the contradiction established in § 4. But in any case, since $\chi_x = 0$ on $r = \lambda$, we have (replacing x_2 by x)

$$\tfrac{1}{4}\Omega^2\lambda^4(x) + 2\int_{\lambda(x)}^{R}\chi_x^2\frac{dr}{r} - \frac{d}{dx}\int_{\lambda(x)}^{R}\chi\chi_x\frac{dr}{r} = 0,$$

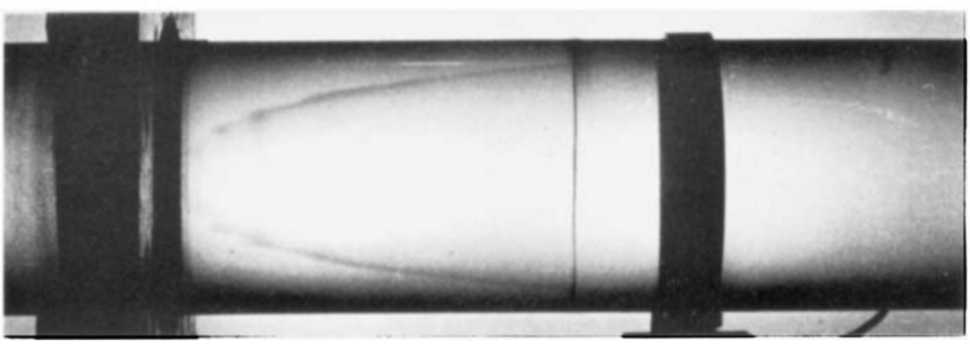
whence

$$\int_{\lambda(x)}^{R}\chi\chi_x\frac{dr}{r} \geqslant \text{const.} + \tfrac{1}{4}\Omega^2\int_0^x \lambda^4(\xi)\,d\xi.$$

By (A 4) the right-hand side $\to \infty$ as $x \to \infty$; so therefore does the left-hand side. That is, velocities are unbounded sufficiently far downstream. But this contradicts the condition (iii).



(*a*)



(*b*)

FIGURE 3 (plate 1). (*a*) Cavity. (*b*) Tellurium dye trace. The open end of the tube is to the right.

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